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Generalising gauge variance for spherically symmetric potentials

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Abstract. We give a group-theoretic construction for a one-parameter family of alternative Lagrangians for the problem of a particle moving under a spherically symmetric potential, recently discovered by Henneaux and Shepley. We also discuss the question of what distinguishes the standard Lagrangian from the others, which is of some importance in relation to the quantisation of the hydrogen atom, for example.

1. Introduction

Henneaux and Shepley have recently shown (1982) that the Lagrangian describing the classical motion of a particle in three-dimensional space moving under the influence of a spherically symmetric potential is not uniquely determined, even allowing for gauge variance (the addition of a total time derivative) and multiplication by a constant. In addition to the usual Lagrangian L given by

$$L(\boldsymbol{x}, \boldsymbol{v}) = \tfrac{1}{2}|\boldsymbol{v}|^2 - V(r) \qquad r = |\boldsymbol{x}|$$

they find a whole family of Lagrangians depending on one undetermined function of two variables. One particularly interesting subfamily is a one-parameter family given by

$$L_\gamma = L + \gamma J / r^2$$

where γ is a real parameter and J is the magnitude of the angular momentum, $J = |\boldsymbol{x} \times \boldsymbol{v}|$.

This study of spherically symmetric systems was motivated by a question by Wigner (1950): do the equations of motion completely determine the commutation relations? As Henneaux and Shepley point out, their results raise some awkward questions relating to quantisation. Since quantisation procedures rely on a Hamiltonian formulation of the classical equations of motion, the existence of alternative Lagrangians for the same system, which do not merely differ by a total time derivative, may lead to alternative, competing quantisations. Thus, as Henneaux and Shepley show, the Lagrangian L_γ for the Kepler problem leads, when $\gamma \neq 0$, to a new quantisation of the hydrogen atom, which has the unfortunate feature that there is no degeneracy of the energy levels. It is clearly desirable to understand as fully as possible how this excessive freedom of choice of Lagrangians arises, in order to formulate some appropriate selection rule which picks out the standard Lagrangian L.

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Henneaux and Shepley obtained their results from a complete analysis of the Helmholtz conditions for spherically symmetric systems. We shall describe a very different approach to the derivation of the Lagrangian L_γ. Our construction is of a group-theoretic kind, and reveals that the freedom to add the term $\gamma J/r^2$ to L is a natural extension of the usual gauge freedom, mediated by the action of the rotation group. As well as describing the construction in particular and in general, we make some remarks about the quantisation problem in the light of our construction.

2. Construction of the Lagrangians

Our method is to exploit the spherical symmetry of the system—in other words, its invariance under the orthogonal group SO(3)—directly. We shall show that by restricting the system to a plane of constant direction of angular momentum in the usual way, then adding a total time derivative to the restricted Lagrangian (the only freedom in the Lagrangian in the two-dimensional case), and finally using the action of SO(3) to reconstruct the entire motion, we obtain the Lagrangians L_γ. We begin by recalling the group action and the corresponding angular momentum reduction in the case of the standard Lagrangian.

The system has the configuration space E_0^3, Euclidean 3-space with the origin removed, and its velocity space TE_0^3 is just $E_0^3 \times E^3$. The group SO(3) acts on TE_0^3 by

$$g(\boldsymbol{x}, \boldsymbol{v}) = (g\boldsymbol{x}, g\boldsymbol{v}) \qquad g \in \mathrm{SO}(3).$$

Any Lagrangian of the form

$$L(\boldsymbol{x}, \boldsymbol{v}) = \tfrac{1}{2}|\boldsymbol{v}|^2 - V(r)$$

is invariant under this action:

$$g^* L = L \qquad \text{for all } g \in \mathrm{SO}(3).$$

It follows that SO(3) is a subgroup of the group of Noether symmetries for the problem, which is in turn a subgroup of the group of Lie symmetries of the dynamical vector field Γ (see Prince (1983b) for this terminology). Thus Γ is invariant under SO(3). Corresponding to this Noether action of SO(3) we have the conservation of angular momentum: the angular momentum function $\boldsymbol{J}: TE_0^3 \to E^3$, where $\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x} \times \boldsymbol{v}$, is conserved by Γ:

$$\Gamma(\boldsymbol{J}) = \boldsymbol{0}.$$

It is moreover equivariant under the action of SO(3):

$$\boldsymbol{J}(g(\boldsymbol{x}, \boldsymbol{v})) = g\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{v}) \qquad \text{for all } g \in \mathrm{SO}(3).$$

The constancy of $\boldsymbol{J}$ allows the familiar classical reduction. One chooses a non-zero element $\bar{\boldsymbol{J}}$ of E^3 and considers the set of points

$$\{(\boldsymbol{x}, \boldsymbol{v}) \in TE_0^3 | \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{v}) = k\bar{\boldsymbol{J}} \text{ for some } k \in R,\ k \neq 0\}$$
$$= \{(\boldsymbol{x}, \boldsymbol{v}) \in TE_0^3 | \boldsymbol{x} \cdot \bar{\boldsymbol{J}} = \boldsymbol{v} \cdot \bar{\boldsymbol{J}} = 0,\ \boldsymbol{x} \times \boldsymbol{v} \neq \boldsymbol{0}\}.$$

This is a submanifold of TE_0^3 of dimension four, which may be identified as an open submanifold of TP, where P is the punctured plane $\boldsymbol{x} \cdot \bar{\boldsymbol{J}} = \boldsymbol{0}$, $\boldsymbol{x} \neq \boldsymbol{0}$. It has two components; we consider the component with $k > 0$. We shall use TP^+ to denote this four-dimensional submanifold. Now Γ is tangent to TP^+, and its restriction $\bar{\Gamma}$ is the Euler–Lagrange field of $\bar{L}$, the restriction of L to TP^+. Moreover SO(2), the isotropy group of $\bar{J}$, acts on P, preserves $\bar{L}$ and is a symmetry of $\bar{\Gamma}$.

The significance of this reduction lies in the fact that the whole dynamics may be recovered from TP^+, $\bar{\Gamma}$ and $\bar{L}$ by the action of SO(3)—at least so far as the open submanifold TE_0^{3+} of TE_0^3 for which $\boldsymbol{x}\times\boldsymbol{v}\neq\boldsymbol{0}$ is concerned. Given any $(\boldsymbol{x},\boldsymbol{v})\in TE_0^{3+}$ there is some $g\in SO(3)$ such that $g(\boldsymbol{x},\boldsymbol{v})\in TP^+$: for there is certainly an element $g\in SO(3)$ such that $g\boldsymbol{J}(\boldsymbol{x},\boldsymbol{v})=k\bar{\boldsymbol{J}}$ for some $k>0$; but then $\boldsymbol{J}(g(\boldsymbol{x},\boldsymbol{v}))=k\bar{\boldsymbol{J}}$ and so $g(\boldsymbol{x},\boldsymbol{v})\in TP^+$. Moreover if g_1 and g_2 both map $(\boldsymbol{x},\boldsymbol{v})$ into TP^+ then $h=g_2g_1^{-1}$ satisfies $h\bar{\boldsymbol{J}}=\bar{\boldsymbol{J}}$; so $h\in SO(2)$ and maps TP^+ to itself. We may therefore define a function $\tilde{L}$ on TE_0^{3+} by

$$\tilde{L}(\boldsymbol{x},\boldsymbol{v})=\bar{L}(g(\boldsymbol{x},\boldsymbol{v}))$$

where g is chosen so that $g(\boldsymbol{x},\boldsymbol{v})\in TP^+$. Since $\bar{L}$ is invariant under the action of SO(2) it does not matter which g is used to map $(\boldsymbol{x},\boldsymbol{v})$ into TP^+, and so $\tilde{L}$ is well defined. It is easy to see that $\tilde{L}$, like L, is invariant under SO(3); and since $\tilde{L}=L=\bar{L}$ on TP$^+$, it follows that $\tilde{L}=L$ everywhere on TE_0^{3+}.

If, now, instead of merely reconstructing L from $\bar{L}$ one modifies $\bar{L}$ first, the resulting L will be a new Lagrangian for Γ, provided that the modification is carried out in a suitable way. Thus one may add to $\bar{L}$ any constant multiple of $\dot{\theta}$ (where r, θ are polar coordinates on the punctured plane P) without affecting the dynamics in P. We set

$$\bar{L}_\gamma=\bar{L}+\gamma\dot{\theta}\qquad\gamma\in R.$$

Then $\bar{L}_\gamma$ is also invariant under the action of SO(2), and we may therefore define a function $\tilde{L}_\gamma$ on TE_0^{3+} by

$$\tilde{L}_\gamma(\boldsymbol{x},\boldsymbol{v})=\bar{L}_\gamma(g(\boldsymbol{x},\boldsymbol{v}))$$

where g is chosen so that $g(\boldsymbol{x},\boldsymbol{v})\in TP^+$, as before. In fact, $\tilde{L}_\gamma$ so defined is just Henneaux and Shepley's L_γ. On TP^+, $J=r^2\dot{\theta}$, and so

$$L_\gamma=\bar{L}+\gamma J/r^2=\bar{L}+\gamma r^2\dot{\theta}/r^2=\bar{L}_\gamma.$$

Thus L_γ and $\tilde{L}_\gamma$ agree on TP^+; and since they are both invariant under SO(3) they must be equal everywhere on TE_0^{3+}.

We have shown that L_γ is obtained by restricting L to TP^+, adding a total time derivative, and using the action of SO(3) to construct a function on TE_0^{3+}. In fact the Lagrangians L_γ of the one-parameter family are essentially the only Lagrangians for Γ which can be obtained in this way: consider again the motion on TP^+. It is well known that the Lagrangian for two-dimensional rotationally symmetric motion is unique up to gauge variance. Thus the most general Lagrangian for $\bar{\Gamma}$ on TP^+ is

$$\bar{L}+\dot{f}$$

where f is a function of r and θ. (We ignore the possibility of multiplying $\bar{L}$ by a constant, since this has the same effect on L.) The new Lagrangian is required to be invariant under SO(2). Now

$$\dot{f}=\frac{\partial f}{\partial r}\dot{r}+\frac{\partial f}{\partial\theta}\dot{\theta}$$

and invariance requires that both $\partial f/\partial r$ and $\partial f/\partial\theta$ should be independent of θ.

However, then

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial \theta}\right)=\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}\right)=0$$

and so $\partial f/\partial\theta$ is actually constant, say $\partial f/\partial\theta=\gamma$. Then $f-\gamma\theta=F$ is a function of r alone, and

$$\dot f=\dot F+\gamma\dot\theta.$$

Thus

$$\bar L+\dot f=\bar L_\gamma+\dot F$$

where F is a function of r alone, and is therefore the restriction to P of a function on E_0^3 invariant under SO(3). The term $\dot F$ will therefore contribute only a gauge term to the Lagrangian on TE_0^{3+} constructed from $\bar L_\gamma+\dot F$. (Actually, the foregoing argument is a little oversimplified: strictly speaking its conclusion should be that if f is a function on P with $\dot f$ invariant under SO(2) then f is a function of r alone, since θ is not a well defined function on P. However it makes sense if the definition of a total time derivative is relaxed slightly. Given any differentiable manifold M, a 1-form ω on M determines a function $\hat\omega$ on TM linear in the fibre coordinates by $\hat\omega(x,v)=\langle v,\omega_x\rangle$. Our argument makes sense if a total time derivative—perhaps gauge term would be more appropriate here—is interpreted to be a function on TM of the form $\hat\omega$ where ω is closed, but not necessarily the exterior derivative of any function f.)

3. Towards a general theory

We now discuss the use of group actions in the construction of Lagrangians in the general case.

Let G be a Lie group of diffeomorphisms of a differentiable manifold X. We shall be concerned with submanifolds of X which have the property of having non-empty intersection with every orbit of the action of G on X. A suitable submanifold with this property will serve as an 'initial data' submanifold for the determination of objects on X which are invariant under the action of G. We say that a submanifold Y of X is a *section of the action of* G if the following two conditions hold.

(i) Every orbit of the action of G on X has non-empty intersection with Y; or in other words, for every $x\in X$ there is some $g\in$ G such that $gx\in Y$.

(ii) For all $x\in X$, if g_1x and g_2x both lie in Y then $g_2g_1^{-1}$ maps the whole of Y into itself.

We call the subgroup H of G which leaves Y invariant its isotropy subgroup. It is certainly true that if $g_1x\in Y$ and $g_2=hg_1$ with $h\in$ H then $g_2x\in Y$ also. Condition (ii) requires that this should be the only possibility.

When a Lie group G acts as a group of diffeomorphisms of a manifold X there is a homomorphism of its Lie algebra $\mathscr{G}$ into the module of vector fields on X, defined as follows: for any $A\in\mathscr{G}$ the corresponding vector field $\tilde A$ on X is the generator of the one-parameter group of diffeomorphisms $\{\exp(tA)\}$. As well as being a homomorphism (that is, being linear and preserving brackets) the map $A\to\tilde A$ satisfies

$$g_*\tilde A=\widetilde{(\mathrm{Ad}\,g)A}\qquad\text{for all } g\in \mathrm{G}.$$

For each point $x \in X$, the set of vectors of the form $\tilde{A}_x$ constitutes a subspace of T_xX, which we denote $\mathcal{G}_x$; it is the tangent space to the orbit of x. If Y is a section of the action of G we shall say that a point $y \in Y$ is *regular* if the subspaces T_yY and $\mathcal{G}_y$ of T_yX together span it.

We specialise now to the case where $X = TM$ is the tangent bundle of a differentiable manifold M. We shall be concerned only with actions on TM induced from actions on M: thus for each $g \in$ G, the diffeomorphism of TM corresponding to g is the tangent map of a diffeomorphism of M. The action of G thus preserves the structure of TM. In particular it preserves the vertical endomorphism S of TM, which defines its almost tangent structure (Crampin 1983a, b). In fact it will be convenient to note that this result holds in a more general situation. Let $\phi: N \to M$ be a smooth map, and $\Phi: TN \to TM$ its tangent map. Let S_N, S_M be the vertical endomorphisms of TN, TM respectively. Then

$$\Phi_* \circ S_N = S_M \circ \Phi_*.$$

For any function λ on TM we define its Cartan 1-form θ_λ by

$$\theta_\lambda = \mathrm{d}\lambda \circ S_M.$$

It follows that $\Phi^*\theta_\lambda$ is a Cartan 1-form on TN, namely the Cartan 1-form corresponding to $\Phi^*\lambda$. Thus when a Lie group G acts on TM and its action is induced from one on M, then if λ is a function on TM invariant under the action, its Cartan 1-form θ_λ is also invariant, in the sense that

$$g^*\theta_\lambda = \theta_\lambda \qquad \text{for all } g \in \mathrm{G}.$$

Furthermore a tangent map $\Phi: TN \to TM$ satisfies

$$\Phi \circ \delta_t^N = \delta_t^M \circ \Phi$$

where $\delta_t^M(\delta_t^N)$ is the one-parameter group of dilations of the fibres of $TM(TN)$. It follows that

$$\Phi_*\Delta^N = \Delta^M|_{\Phi(N)}$$

where $\Delta^M(\Delta^N)$ is the dilation vector field on $TM(TN)$. In the case of a group action (or any diffeomorphism) on M the dilation vector field is invariant.

Suppose now that Γ is a second-order differential equation field on TM, and that there is an embedded submanifold N of M such that Γ is tangent to TN. Suppose that the Lie group G acts as a group of Lie symmetries of Γ: in other words, G acts on TM, its action being induced from one on M, and for each $g \in$ G

$$g_*\Gamma = \Gamma.$$

Suppose that TN is a section of the action of G, and let its isotropy group be H. We shall show that the construction of the previous section can be generalised to apply in this situation, but only so as to apply to the regular points of TN. Let TN^+ be the set of regular points of TN, and TM^+ its image in TM under the action of G. We shall use $\bar{\Gamma}$ to denote the restriction of Γ to TN^+. Suppose that there is a Lagrangian $\bar{L}$ for $\bar{\Gamma}$ (so $\bar{L}$ is a function on TN^+); suppose also that $\bar{L}$ is invariant under H. Define a function L on TM^+ by

$$L(p) = \bar{L}(gp)$$

where $p \in TM^+$ and $g \in$ G is such that $gp \in TN^+$; the invariance of $\bar{L}$ ensures that L

is well defined. We show that provided one additional condition is satisfied, L is a Lagrangian for Γ on TM^+.

Theorem. Under the conditions described above, if for every $A \in \mathscr{G}$ the restriction to TN^+ of the function $\langle \tilde{A}, \theta_L \rangle$ (where θ_L is the Cartan 1-form of L) is conserved by $\bar{\Gamma}$ then L is a Lagrangian for Γ on TM^+; it is invariant under G, and is the unique Lagrangian for Γ invariant under G which agrees with $\bar{L}$ on TN^+.

Proof. To show that L is a Lagrangian for Γ on TM^+ we must show that

$$i_\Gamma \, \mathrm{d}\theta_L = -\mathrm{d}E_L$$

where

$$E_L = \Delta L - L$$

is the energy, Δ being the dilation field on TM. Now L is invariant under the action of G: for if $p \in TM^+$ and $g' \in \mathrm{G}$ is such that $g'p \in TN^+$, then for $g \in \mathrm{G}$ the group element $g'g^{-1}$ maps gp into TN^+ and so

$$L(gp) = \bar{L}((g'g^{-1})(gp)) = \bar{L}(g'p) = L(p).$$

The action of G therefore preserves θ_L; and since it leaves Δ invariant it also preserves E_L. It preserves Γ by assumption. Thus the 1-form $i_\Gamma \, \mathrm{d}\theta_L + \mathrm{d}E_L$ is invariant under the action of G, and so it is sufficient to show that it vanishes at every point of TN^+ in order to show that it vanishes everywhere on TM^+. Our discussion above about the behaviour of Cartan 1-forms and dilation fields, applied to the embedding of TN^+ in TM^+, shows that the restriction of $i_\Gamma \, \mathrm{d}\theta_L + \mathrm{d}E_L$ to TN^+ is just $i_{\bar{\Gamma}} \, \mathrm{d}\theta_{\bar{L}} + \mathrm{d}E_{\bar{L}}$; 'restriction' here means restriction of the arguments of the 1-form to be tangent to TN^+. But this restriction vanishes since, by assumption, $\bar{L}$ is a Lagrangian for $\bar{\Gamma}$. It remains to be shown that $i_\Gamma \, \mathrm{d}\theta_L + \mathrm{d}E_L$ vanishes at points of TN^+ when its arguments are transverse to TN^+. Since TN^+ consists of regular points, it is enough to consider the evaluation of the 1-form on a vector field of the form $\tilde{A}$, where $A \in \mathscr{G}$. Now since the action of G on TM is induced from an action on M, and since L is invariant, we have

$$\mathscr{L}_{\tilde{A}} S = 0 \qquad [\tilde{A}, \Delta] = 0 \qquad \tilde{A} L = 0.$$

It follows that

$$\tilde{A} E_L = 0 \qquad \mathscr{L}_{\tilde{A}} \theta_L = 0.$$

From the second of these,

$$i_{\tilde{A}} \, \mathrm{d}\theta_L = -\mathrm{d}\langle \tilde{A}, \theta_L \rangle.$$

Thus

$$\langle \tilde{A}, i_\Gamma \, \mathrm{d}\theta_L + \mathrm{d}E_L \rangle = -\langle \Gamma, i_{\tilde{A}} \, \mathrm{d}\theta_L \rangle + \tilde{A} E_L = \Gamma \langle \tilde{A}, \theta_L \rangle$$

and on TN^+ this vanishes since by assumption $\langle \tilde{A}, \theta_L \rangle$ is conserved by $\bar{\Gamma}$. It follows that

$$i_\Gamma \, \mathrm{d}\theta_L = -\mathrm{d}E_L$$

on TM^+, as required. Now any function on TM^+ invariant under the action of G is determined by its values on a section of the action; so L is the unique G-invariant Lagrangian which agrees with $\bar{L}$ on TN^+.

This result implies that in the case of a spherically symmetric system the only spherically symmetric Lagrangians on TE_0^{3+} are those of the one-parameter family L_γ. We can now explain the significance of the condition $\boldsymbol{x}\times\boldsymbol{v}\neq\boldsymbol{0}$: it identifies the regular points of TP. For suppose that coordinates (x^1, x^2, x^3) are chosen in E^3 with p as the plane $x^3=0$. If (v^1, v^2, v^3) are the corresponding fibre coordinates in TE^3 then the generators of rotations about the x^1 and x^2 axes induce on TP the vector fields

$$x^2\frac{\partial}{\partial x^3}+v^2\frac{\partial}{\partial v^3} \qquad \text{and} \qquad -x^1\frac{\partial}{\partial x^3}-v^1\frac{\partial}{\partial v^3}.$$

These will be linearly independent except when

$$x^2v^1-x^1v^2=0,$$

that is, except when

$$\boldsymbol{x}\times\boldsymbol{v}=(x^1, x^2, 0)\times(v^1, v^2, 0)=\boldsymbol{0}.$$

Our construction gives a Lagrangian only on TE_0^{3+}, the open submanifold of TE_0^3 obtained from TP^+ by the action of SO(3). Note that this is a significant restriction, since the function J is not smooth at $J=0$; only in the case of the standard Lagrangian $(\gamma=0)$ is it possible to extend L_γ smoothly to the whole of TE_0^3.

The condition on $\langle\tilde{A}, \theta_L\rangle$ arises from the fact that when G acts in the manner described, this quantity is the constant of the motion associated with A by Noether's theorem. Thus it is certainly to be expected that $\langle\tilde{A}, \theta_L\rangle$ will be conserved by $\bar{\Gamma}$ on TN^+; and this proves to be a sufficient assumption to allow one to transfer the information on TN^+ into the surrounding space. Of course when A belongs to the Lie algebra of H the constancy of $\langle\tilde{A}, \theta_L\rangle$ on TN^+ is automatic. In the case of a spherically symmetric system this quantity is actually identically zero on TP^+ when A is a generator of rotations about an axis in P, since it is a multiple of the component of angular momentum in the direction of that axis.

When (as in the case of a spherically symmetric potential) one Lagrangian is already known and is invariant under a Lie group G acting as Lie symmetries, it may be possible to use the corresponding constants of the motion to construct a section of the action. For each $A\in\mathscr{G}$ we set

$$\mathscr{J}_A=\langle\tilde{A}, \theta_L\rangle$$

where L is the known Lagrangian; then $\mathscr{J}_A$ is a function on TM and is a constant of the motion. Moreover $\mathscr{J}_A$ depends linearly on A. Thus for each $p\in TM$ the map $\mathscr{G}\to R$ by $A\to\mathscr{J}_A(p)$ is linear, and therefore defines an element of $\mathscr{G}^*$, the vector space dual to $\mathscr{G}$. We may therefore define a map $\mathscr{J}: TM\to\mathscr{G}^*$ by

$$\langle A, \mathscr{J}(p)\rangle=\mathscr{J}_A(p)=\langle\tilde{A}, \theta_L\rangle_p \qquad \text{for all } A\in\mathscr{G}.$$

From the fact that $g_*\tilde{A}=\widetilde{(\mathrm{Ad}\, g)A}$ it follows that $\mathscr{J}$ is equivariant under the action of G on TM and its co-adjoint action on $\mathscr{G}^*$:

$$\mathscr{J}(gp)=(\mathrm{Ad}\, g^{-1})^*\mathscr{J}(p).$$

The map $\mathscr{J}$ is called the momentum map associated with the action of G (Weinstein 1977, lecture 4).

Now suppose that $\mathscr{S}$ is a subspace of $\mathscr{G}^*$ which is a section of the co-adjoint action. Suppose that, as a homomorphism of G into the automorphism group of $\mathscr{G}$, Ad is injective. Then the inverse image $\mathscr{J}^{-1}(\mathscr{S}) \subset TM$ is a section of the action of G on TM. For given any $p \in TM$ there is some $g \in \mathrm{G}$ such that $(\mathrm{Ad}\, g^{-1})^* \mathscr{J}(p) \in \mathscr{S}$; but then $\mathscr{J}(gp) \in \mathscr{S}$ and so $gp \in \mathscr{J}^{-1}(\mathscr{S})$. If both $g_1 p \in \mathscr{J}^{-1}(\mathscr{S})$ and $g_2 p \in \mathscr{J}^{-1}(\mathscr{S})$ then both $(\mathrm{Ad}\, g_1^{-1})^* \mathscr{J}(p) \in \mathscr{S}$ and $(\mathrm{Ad}\, g_2^{-1})^* \mathscr{J}(p) \in \mathscr{S}$, and therefore

$$(\mathrm{Ad}\, g_2^{-1})^* (\mathrm{Ad}\, g_1)^* = (\mathrm{Ad}\, h^{-1})^*$$

for some $h \in \mathrm{H}$, the isotropy subgroup of $\mathscr{S}$. But then

$$\mathrm{Ad}(g_2 g_1^{-1}) = \mathrm{Ad}\, h \qquad \text{and so} \qquad g_2 g_1^{-1} = h.$$

(The injectivity assumption covers the SO(3) case; a more general result could be obtained by factoring out the kernel of Ad, which is the centre of G, from the co-adjoint action.)

4. The quantisation problem

The problem that the non-uniqueness of Lagrangians raises for quantisation is that different Lagrangians may lead to different quantum theories for the same system. One needs to know, therefore, how to pick out at the start the Lagrangian which will give the correct quantum mechanics. This cannot be done on a phenomenological basis at the classical level. However, our discussion above does identify a couple of factors which single out the standard Lagrangian in the case of a spherically symmetric system. In the first place it would seem natural to require that the Lagrangian itself should be spherically symmetric: this forces it to be of the form L_γ. However if we then require the Lagrangian to be smooth all over TE_0^3 and not just on TE_0^{3+}—in other words if we require it to govern radial motion—then the standard Lagrangian is the only possibility. There is some prospect that similar considerations may apply in other highly symmetric situations; and it seems likely that a high degree of symmetry is necessary for there to be an oversupply of Lagrangians—Henneaux and Shepley (1982) have pointed out that the introduction of some anisotropy destroys this feature of spherically symmetric systems.

In the particular case of the hydrogen atom there are additional considerations. Henneaux and Shepley showed that when quantisation of the hydrogen atom is carried out starting from a non-standard Lagrangian there is no longer the usual degeneracy of the energy levels. Now the degeneracy of the energy levels is associated with the so-called hidden symmetries of the system. In the classical picture these hidden symmetries are associated with the constancy of the Runge-Lenz vector. There are two ways of relating the Runge-Lenz vector to symmetries of the system. One is through the Noether-Cartan theorem, which associates a symmetry vector field with every conserved quantity. In the case of the Runge-Lenz vector this procedure, used with the standard Lagrangian, yields symmetry vector fields on TE_0^3 which are not Lie symmetries but are derived from the rank-2 Killing tensors of the Euclidean metric in a natural way (Crampin 1984). This natural geometrical structure of the hidden symmetries does not appear to survive the modification of the Lagrangian. The second approach to the Runge-Lenz vector is to generate it from a Lie symmetry of the dynamical vector field in a way which is essentially independent of the Cartan structure (Prince and Eliezer 1981). The Lie symmetry in question is one associated with Kepler's third law; its generator is the vector field Z on evolution space $R \times TE_0^3$ (R for time) given in the usual coordinates by

$$Z = t\frac{\partial}{\partial t} + \frac{2}{3}x^a\frac{\partial}{\partial x^a} - \frac{1}{3}v^a\frac{\partial}{\partial v^a}$$

(Prince 1983a). According to the theory developed by Prince (Prince 1983b, Crampin and Prince 1985) such a Lie symmetry vector field which does not preserve a Lagrangian may be used to generate a new one by Lie differentiation. Once again the standard Lagrangian occupies a distinguished position: it is in effect a fixed point of the action of Z in this construction. We find that

$$\mathscr{L}_Z L_\gamma = -\tfrac{2}{3}L_\gamma - \tfrac{1}{3}\gamma J/r^2 = -\tfrac{2}{3}L_{3\gamma/2}.$$

Thus when $\gamma = 0$ the Lagrangian changes only by multiplication by a constant; otherwise, the action of Z produces a change of parameter value as well.

In this aspect of the problem it is the relation between Lie and Cartan symmetries of the system which is at issue, and this again depends on the choice of Lagrangian in a rather complicated way. It seems that in the case of the hydrogen atom the interaction between Lie and Cartan symmetries is simplest for the standard Lagrangian. However, these are topics for further investigation.

Finally our construction, which derives the alternative spherically symmetric Lagrangians from gauge variance via the group action, holds out some hope that it may be possible to modify the quantisation procedure (which after all ignores gauge terms) when a group of Lie symmetries acts so as to ignore all deviations from the standard Lagrangian.

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